Feedback stabilization of the Boussinesq system

Ruchi Sandilya

Joint Work: Praveen Chandrashekar, Mythily Ramaswamy (TIFR CAM)
Jean-Pierre Raymond (Institut de Mathématiques)

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Outline

Motivation

- Boussinesq equations consist of the Navier–Stokes equations coupled to the convection–diffusion equation for temperature
- Frequently used in modeling, designing, and controlling energy-efficient building systems
- Building efficiency is essential to meet national energy and environmental challenges
- Boussinesq systems are unstable for certain values of its parameters.
- We developed efficient feedback control strategies to stabilize the system and optimize energy use in the building
- Stability analysis is necessary to understand the flow transition from stable to turbulent regimes

The model problem

Navier-Stokes-Boussinesq system

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \rho &= \frac{2}{\mathrm{Re}} \nabla \cdot \varepsilon(\boldsymbol{u}) + \frac{\mathrm{Gr}}{\mathrm{Re}^2} \tau \boldsymbol{e}_2 \\
\frac{\partial \tau}{\partial t} + \boldsymbol{u} \cdot \nabla \tau &= \frac{1}{\mathrm{RePr}} \Delta \tau + \phi_s \\
\nabla \cdot \boldsymbol{u} &= 0
\end{aligned}
$$

$\boldsymbol{u}(x,t):$ velocity $\quad \rho(x,t):$ pressure $\quad \tau(x,t):$ temperature

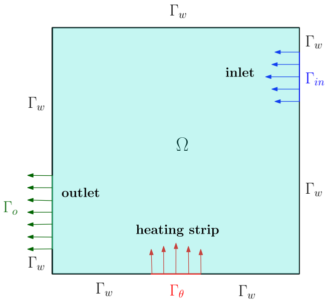
$\boldsymbol{e}_2 = (0,1), \quad \phi_s = 7\sin(2\pi x)\cos(2\pi y)$

$\varepsilon(\boldsymbol{u}) = \frac{1}{2}[\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top]$: strain tensor

$$\mathrm{Re} = 100, \quad \mathrm{Gr} = \frac{\mathrm{Re}^2}{0.9} \approx 11111.1, \quad \mathrm{Pr} = 0.7$$

Domain and its boundary

We consider domain $\Omega = [0,1] \times [0,1]$, with boundary $\Gamma = \Gamma_{in} \cup \Gamma_o \cup \Gamma_\theta \cup \Gamma_w$.



$$\Gamma_{in} = \{1\} \times [0.7, 0.9]$$
$$\Gamma_o = \{0\} \times [0.1, 0.4]$$
$$\Gamma_\theta = [0.4, 0.6] \times \{0\}$$
$$\Gamma_w = \Gamma \setminus (\Gamma_{in} \cup \Gamma_o \cup \Gamma_\theta)$$

Boundary conditions

$$\boldsymbol{u} = 0 \quad \text{on} \quad (\Gamma_{in} \cup \Gamma_\theta \cup \Gamma_w) \times (0,T)$$
$$-\rho\boldsymbol{n} + \frac{2}{\mathrm{Re}}\varepsilon(\boldsymbol{u})\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_o \times (0,T)$$
$$\tau = 0 \quad \text{on} \quad (\Gamma_{in} \cup \Gamma_w) \times (0,T)$$
$$\frac{1}{\mathrm{RePr}}\frac{\partial \tau}{\partial n} = 0 \quad \text{on} \quad (\Gamma_\theta \cup \Gamma_o) \times (0,T)$$

Initial conditions

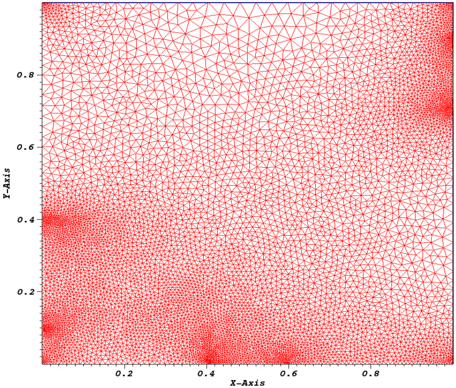
$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \tau(0) = \tau_0 \quad \text{in} \quad \Omega.$$

Stationary problem

Let $(\boldsymbol{u}_s, \tau_s, \rho_s)$ be a solution of stationary problem

$$
\begin{aligned}
\boldsymbol{u}_s \cdot \nabla \boldsymbol{u}_s + \nabla \rho_s &= \frac{2}{\mathrm{Re}} \nabla \cdot \varepsilon(\boldsymbol{u}_s) + \frac{\mathrm{Gr}}{\mathrm{Re}^2} \tau_s \boldsymbol{e}_2 \quad \text{in} \quad \Omega \\
\boldsymbol{u}_s \cdot \nabla \tau_s &= \frac{1}{\mathrm{RePr}} \Delta \tau_s + \phi_s \quad \text{in} \quad \Omega \\
\nabla \cdot \boldsymbol{u}_s &= 0 \quad \text{in} \quad \Omega \\
\boldsymbol{u}_s &= 0 \quad \text{on} \quad \Gamma_{in} \cup \Gamma_\theta \cup \Gamma_w \\
-\rho_s \boldsymbol{n} + \frac{2}{\mathrm{Re}} \varepsilon(\boldsymbol{u}_s) \boldsymbol{n} &= 0 \quad \text{on} \quad \Gamma_o \\
\tau_s &= 0 \quad \text{on} \quad \Gamma_{in} \cup \Gamma_w \\
\frac{1}{\mathrm{RePr}} \frac{\partial \tau_s}{\partial n} &= 0 \quad \text{on} \quad \Gamma_\theta \cup \Gamma_o
\end{aligned}
$$

Locally refined mesh



The mesh is refined near corners and at inlet/outlet portions of the boundary

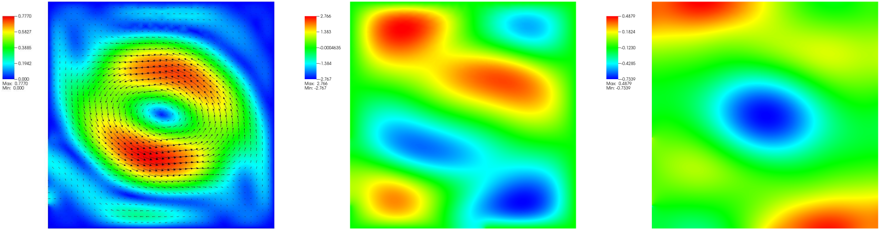
Number of nodes = 7400, Number of degrees of freedom = 99991

Stationary solution: $\mathbf{P}_2 - P_2 - P_1$ FEM, refined mesh

- Solving steady Navier-Stokes at $\mathrm{Re} = 100$ is numerically unstable
- Solve the steady Navier-Stokes on a sequence of Reynolds numbers

$$50 \to 60 \to 70 \to 80 \to 85 \to 90 \to 95 \to 100$$

using solution from previous Re as initial guess in the Newton method.



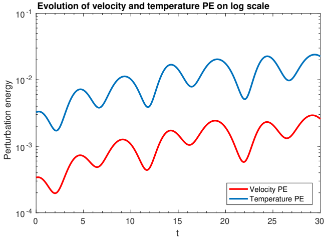
(a) Velocity (b) Temperature (c) Pressure

Stability of stationary solution

Add a small perturbation to stationary state

at the initial condition

$$\begin{bmatrix} \boldsymbol{u}_0 \\ \tau_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_s \\ \tau_s \end{bmatrix} + \epsilon \begin{bmatrix} \boldsymbol{u}_e \\ \tau_e \end{bmatrix}$$

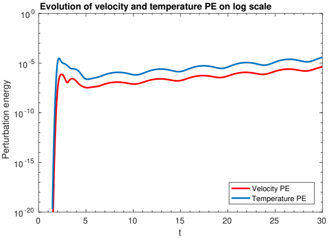


$\epsilon = 0.1$

as a source term

$$\frac{\partial \tau}{\partial t} + \boldsymbol{u} \cdot \nabla \tau = \frac{1}{\mathrm{RePr}} \Delta \tau + \phi_s + \phi$$

$$\phi = \epsilon \exp(-50(t-2)^2) \sin(2\pi x) \cos(2\pi y)$$



$\epsilon = 0.1$

Energy in perturbations for velocity and temperature

$$E_{\boldsymbol{u}} = \frac{1}{2} \int_{\Omega} |\boldsymbol{u} - \boldsymbol{u}_s|^2 \mathrm{d}x, \quad E_\tau = \frac{1}{2} \int_{\Omega} |\tau - \tau_s|^2 \mathrm{d}x.$$

Boundary controls

To acheive stabilization, we apply velocity and temperature controls on Γ_{in}

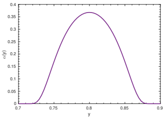
$$\boldsymbol{u} = 0 \quad \text{on} \quad (\Gamma_\theta \cup \Gamma_w) \times (0,T), \quad \boldsymbol{u} = (u_c, 0) \quad \text{on} \quad \Gamma_{in} \times (0,T)$$

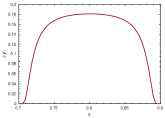
$$\sigma_{\boldsymbol{u}} = -\rho \boldsymbol{n} + \frac{2}{\text{Re}} \varepsilon(\boldsymbol{u}) \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_o \times (0,T)$$

$$\tau = \tau_c \quad \text{on} \quad \Gamma_{in} \times (0,T), \quad \tau = 0 \quad \text{on} \quad \Gamma_w \times (0,T)$$

$$\sigma_\tau = \frac{1}{\text{RePr}} \frac{\partial \tau}{\partial n} = 0 \quad \text{on} \quad (\Gamma_\theta \cup \Gamma_o) \times (0,T)$$

$u_c = f_1 \alpha(y), \quad \tau_c = f_2 \beta(y), \quad (f_1, f_2)$: control variables





$$\alpha(y) = \exp\left(-\frac{0.0001}{[(0.7-y)(0.9-y)]^2}\right), \quad \beta(y) = 0.2 \exp\left(-\frac{0.00001}{[(0.7-y)(0.9-y)]^2}\right)$$

Finite dim feedback control approach

- Stationary state is unstable
  - Small perturbation take the state away
- Aim: apply control to drive state towards stationary state
- Model perturbation z by linearization around stationary state

$$M\frac{\mathrm{d}z}{\mathrm{d}t} = Az + Bf$$

- Determine control by feedback law: $f = -\tilde{K}z$ to achieve

$$\|z(t)\| \longrightarrow 0, \quad \text{as} \quad t \longrightarrow \infty$$

- Determine feedback matrix $\tilde{K}$ from associated ARE to achieve $A - B\tilde{K}$ stable (real$(\lambda) < 0$)
- Use linear feedback law $f = -\tilde{K}z$ to the nonlinear model to study its ability to stabilize the flow

Linearized system

Let $(\boldsymbol{v}, \theta, p) = (\boldsymbol{u}, \tau, \rho) - (\boldsymbol{u}_s, \tau_s, \rho_s)$. The linearized system around $(\boldsymbol{u}_s, \tau_s, \rho_s)$ is

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{u}_s \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{u}_s + \nabla p = \frac{2}{\mathrm{Re}} \nabla \cdot \varepsilon(\boldsymbol{v}) + \frac{\mathrm{Gr}}{\mathrm{Re}^2} \theta \boldsymbol{e}_2 \quad \text{in} \quad \Omega \times (0, T)$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u}_s \cdot \nabla \theta + \boldsymbol{v} \cdot \nabla \tau_s = \frac{1}{\mathrm{RePr}} \Delta \theta \quad \text{in} \quad \Omega \times (0, T)$$

$$\nabla \cdot \boldsymbol{v} = 0 \quad \text{in} \quad \Omega \times (0, T)$$

Boundary condition

$$\boldsymbol{v} = 0 \quad \text{on} \quad (\Gamma_\theta \cup \Gamma_w) \times (0, T)$$

$$\boldsymbol{v} = \boldsymbol{u}_c \quad \text{on} \quad \Gamma_{in} \times (0, T)$$

$$\sigma_{\boldsymbol{v}} = -p\boldsymbol{n} + \frac{2}{\mathrm{Re}} \varepsilon(\boldsymbol{v})\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_o \times (0, T)$$

$$\theta = \tau_c \quad \text{on} \quad \Gamma_{in} \times (0, T)$$

$$\theta = 0 \quad \text{on} \quad \Gamma_w \times (0, T)$$

$$\sigma_\theta = \frac{1}{\mathrm{RePr}} \frac{\partial \theta}{\partial n} = 0 \quad \text{on} \quad (\Gamma_\theta \cup \Gamma_o) \times (0, T)$$

Initial condition

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 = \boldsymbol{u}_0 - \boldsymbol{u}_s,$$

$$\theta(0) = \theta_0 = \tau_0 - \tau_s \quad \text{in} \quad \Omega$$

State space model

Semidiscrete linearized system (matrix form) using $\mathbf{P}_2 - P_2 - P_1$ FEM

$$M_{\boldsymbol{v}\boldsymbol{v}} \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = A_{\boldsymbol{v}\boldsymbol{v}}\boldsymbol{v} + A_{\boldsymbol{v}\theta}\theta + A_{\boldsymbol{v}p}p + A_{\boldsymbol{v}\sigma_{\boldsymbol{v}}}\sigma_{\boldsymbol{v}}$$

$$M_{\theta\theta} \frac{\mathrm{d}\theta}{\mathrm{d}t} = A_{\theta\boldsymbol{v}}\boldsymbol{v} + A_{\theta\theta}\theta + A_{\theta\sigma_\theta}\sigma_\theta$$

$$0 = A_{\boldsymbol{v}p}^\top \boldsymbol{v}, \quad 0 = A_{\boldsymbol{v}\sigma_{\boldsymbol{v}}}^\top \boldsymbol{v} - B_{\boldsymbol{v}_{in}} f_1, \quad 0 = A_{\theta\sigma_\theta}^\top \theta - B_{\theta_{in}} f_2$$

State space representation

$$M \frac{\mathrm{d}z}{\mathrm{d}t} = Az + Bf$$

Differential and algebraic parts

$$M_{yy} \frac{\mathrm{d}y}{\mathrm{d}t} = A_{yy}y + A_{yq}q$$

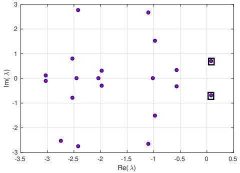
$$0 = A_{yq}^\top y - B_{qf} f$$

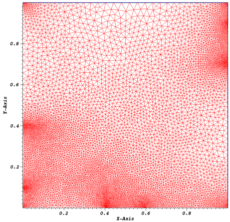
$$y = \begin{bmatrix} \boldsymbol{v} \\ \theta \end{bmatrix}, \ q = \begin{bmatrix} p \\ \sigma_{\boldsymbol{v}} \\ \sigma_\theta \end{bmatrix}, \ f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \ M_{yy} = \begin{bmatrix} M_{\boldsymbol{v}\boldsymbol{v}} & 0 \\ 0 & M_{\theta\theta} \end{bmatrix}, \ A_{yy} = \begin{bmatrix} A_{\boldsymbol{v}\boldsymbol{v}} & A_{\boldsymbol{v}\theta} \\ A_{\theta\boldsymbol{v}} & A_{\theta\theta} \end{bmatrix}$$

$$A_{yq} = \begin{bmatrix} A_{\boldsymbol{v}p} & A_{\boldsymbol{v}\sigma_{\boldsymbol{v}}} & 0 \\ 0 & 0 & A_{\theta\sigma_\theta} \end{bmatrix}, \ B_{qf} = \begin{bmatrix} 0 & 0 \\ B_{\boldsymbol{v}_{in}} & 0 \\ 0 & B_{\theta_{in}} \end{bmatrix}$$

Eigenvalues of linearized operator

Eigenvalues of linearized operator from FEM





h	λ_1, λ_2
1/50	0.0758321220 $\pm$ 0.6947989626i
1/100	0.0802859545 $\pm$ 0.6944522731i
1/150	0.0817657155 $\pm$ 0.6943641588i
1/200	0.0825035403 $\pm$ 0.6943187463i
locally refined	0.0834556781 $\pm$ 0.6945056097i
Extrapolation	0.0847026210 $\pm$ 0.6942351040i

- The spectrum is characterized by two complex conjugated eigenvalues.
- The two unstable eigenvalues are boxed.
- Positive real parts signifying that the linearized problem is unstable to small perturbations.
- Locally refined mesh predicts eigenvalues to good accuracy with less computational cost.
- The number of dofs with 200×200 points on boundary is 522804 while for locally refined mesh is 99991.

Summary of computing feedback matrix

- Determine control by feedback law $f = -\tilde{K}z$ such that $A - B\tilde{K}$ is stable (real$(\lambda) < 0$)
- Form the matrices A_{yy}, A_{yq}, M_{yy}, B_{qf}, A, B, M
- Compute $B_{yq} = A_{yy}M_{yy}^{-1}A_{yq}(A_{yq}^{\top}M_{yy}^{-1}A_{yq})^{-1}B_{qf}$
- Define $A_u = \Xi_u^{\top}A_{yy}E_u$, $B_u = \Xi_u^{\top}B_{yq}$
- Solve Riccati equation for π

$$A_u^{\top}\pi + \pi A_u - \pi B_u B_u^{\top}\pi = 0$$

- Compute feedback operator: $\tilde{K} = (B^{\top}\Xi)\pi(\Xi_u^{\top}M_{yy})$

P. Chandrashekar, M. Ramaswamy, J.P. Raymond, R. Sandilya, Computers & Mathematics with Applications, 2021.

L. Thevenet, PhD Thesis, 2009.

Solving the Boussinesq equations

After spatial discretization: System of nonlinear ODE

$$\hat{M}\frac{\mathrm{d}z}{\mathrm{d}t} = N(z;f), \quad z = \begin{bmatrix} y \\ p \end{bmatrix}, \quad y = \begin{bmatrix} \boldsymbol{u} \\ \tau \end{bmatrix}, \quad \hat{M} = \begin{bmatrix} M_{yy} & 0 \\ 0 & 0 \end{bmatrix}$$

Time Solver: Classical backward difference formula with time step

$$\Delta t = \min_{K\in\mathcal{T}_h} \frac{h_K}{\boldsymbol{u}_K}, \quad \boldsymbol{u}_K = \frac{1}{|K|}\int_K \|u\|\,\mathrm{d}x$$

First time step: BDF1 (Backward Euler)

$$\hat{M}\frac{z^1 - z^0}{\Delta t^1} = N(z^1; f^1), \quad f^1 = \tilde{K}(y^0 - y_s) \quad \text{on} \quad \Gamma_{in}$$

Second time step onwards: BDF2 for $n = 2, 3, \cdots$

$$\frac{1}{\Delta t^n}\hat{M}\left(\frac{2r^n+1}{r^n+1}z^n - (r^n+1)z^{n-1} + \frac{(r^n)^2}{r^n+1}z^{n-2}\right) = N(z^n; f^n),\ r^n = \frac{\Delta t^n}{\Delta t^{n-1}}$$

$$f^n = \tilde{K}(y^* - y_s) \quad \text{on} \quad \Gamma_{in}, \quad y^* = y^{n-2} + (r^{n-1}+1)[y^{n-1} - y^{n-2}]$$

Nonlinear systems are solved using Newton Method and UMFPACK (LU solver).

Control Strategies

- Numerical results for both the cases
  - Distributed perturbation

$$\phi = \epsilon \exp(-50(t-2)^2) \sin(2\pi x) \cos(2\pi y)$$

  - Initial perturbation

$$\begin{bmatrix} \boldsymbol{u}_0 \\ \tau_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_s \\ \tau_s \end{bmatrix} + \epsilon \begin{bmatrix} \boldsymbol{u}_e \\ \tau_e \end{bmatrix}$$

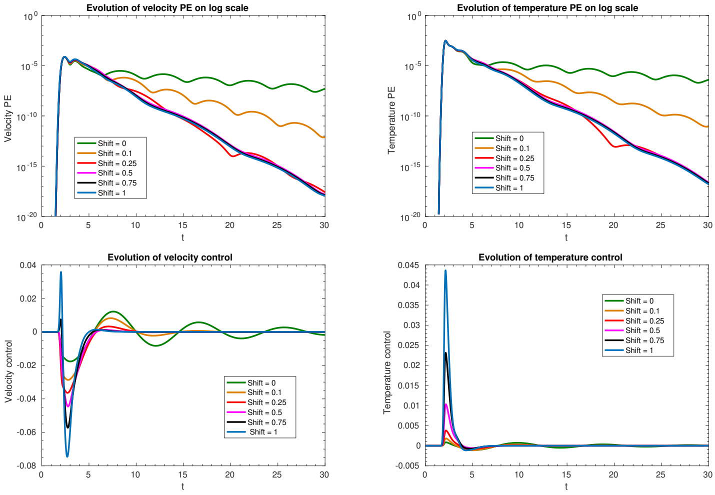
- Obtain feedback by solving 2×2 Riccati equation for different shifts

$$A_u^\top \pi + \pi A_u - \pi B_u B_u^\top \pi = 0, \quad A_u = \Xi_u^\top A_{yy} E_u + \omega I$$

- Test the role of shift parameter ω by shifting first pair of eigenvalues
- Test the role of amplitude of perturbation ϵ

Numerical results: distributed perturbation

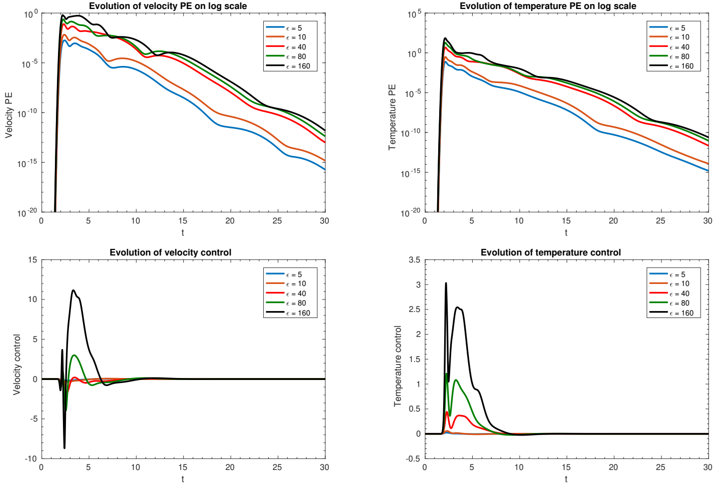
Role of shift parameter: $\omega = \{0, 0.1, 0.25, 0.5, 0.75, 1\}$



$$\epsilon = 1$$

Numerical results: distributed perturbation

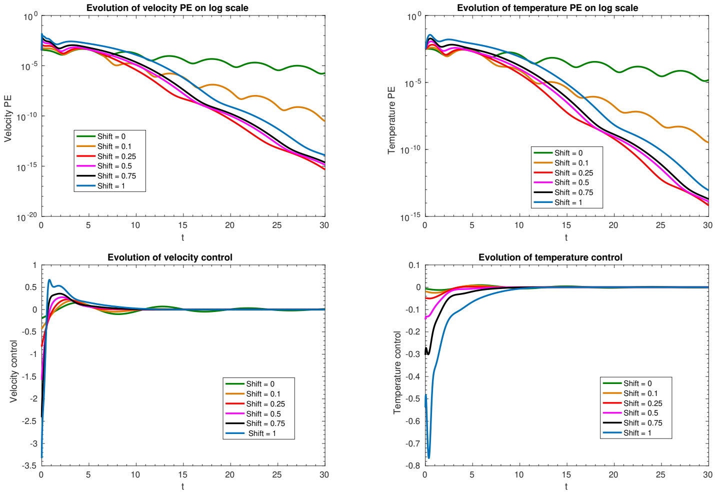
Role of amplitude of perturbation $\epsilon = \{5, 10, 40, 80, 160\}$



$$\omega = 0.25$$

Numerical results: initial perturbation

Role of shift parameter: $\omega = \{0, 0.1, 0.25, 0.5, 0.75, 1\}$



$$\epsilon = 0.1$$

Control with ramp

Smooth ramp function

- Applying large control suddenly at initial time is undesirable.
- Introduce control smoothly over time via smooth ramp function $f(t)$

$$f(t) = g\left(\frac{t-1}{2}\right), \quad g(s) = \begin{cases} 0 & \text{if} \quad s < -1 \\ 0.5 + s(0.9375 - s^2(0.625 - 0.1875s^2)) & \text{if} \quad -1 \le s \le 1 \\ 1 & \text{if} \quad s > 1 \end{cases}$$

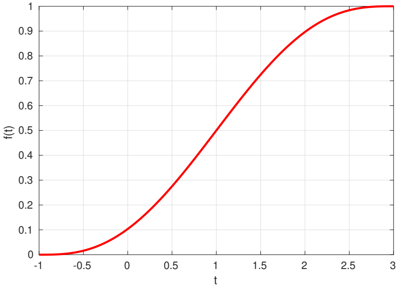
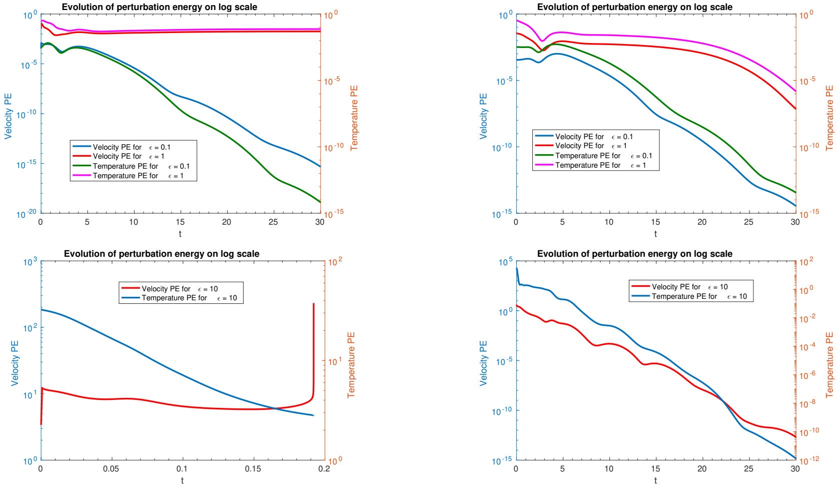


Figure: A plot of function $f(t)$.

Numerical results: initial perturbation

Role of amplitude of perturbation $\epsilon = \{0.1, 1, 10\}$, $\omega = 0.25$



Without ramp

With ramp

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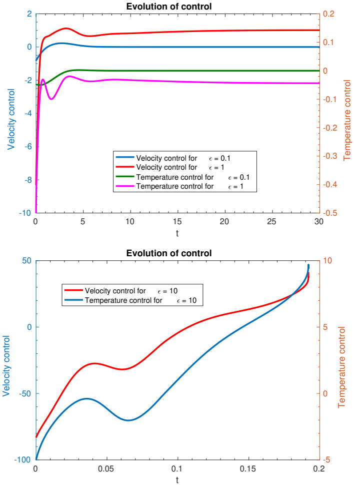
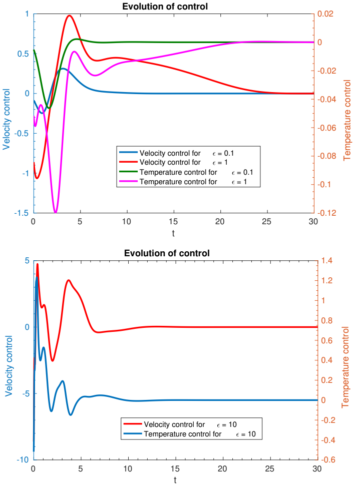
Numerical results: initial perturbation

Role of amplitude of perturbation $\epsilon = \{0.1, 1, 10\}$, $\omega = 0.25$



Without ramp



With ramp

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Summary

- Unstable stationary solution
- Controls at inflow boundary
- Linearized system around the unstable stationary solution
- Unstable eigenvalues of linearized Boussinesq system
- Linear feedback law
- Feedback stabilization with different control strategies
- Numerical results
- Future perspective
  - More efficient strategies for stabilizing the Boussinesq system
  - Better numerics
  - More general (parametrized) Boussinesq system

Collaborators

Joint work with:

Praveen Chandrashekar
TIFR CAM

Mythily Ramaswamy
TIFR CAM

Jean Pierre Raymond
Institut de Mathématiques

P. Chandrashekar, M. Ramaswamy, J. P. Raymond, and R. Sandilya, "Numerical stabilization of the Boussinesq system using boundary feedback control", Computers & Mathematics with Applications, vol. 89, pp. 163–183, 2021.